moment coefficient, C is the chord, I_y is the moment of inertia, γ is the flight-path angle, q is the pitch rate, α is the angle of attack, θ is the attitude angle, and H is the altitude.

The problem is to bring the system to follow the programmed trajectory given by the attitude angle $\theta_p(t)$. Then the model state x_k will contain the tracking error and its derivatives with respect to t. This nonlinear model can be linearized around the actual error and can be brought into a controllable and observable parametrized state space model of the form

$$x(k+1) = A(\xi_k)x(k) + B(\xi_k)u(k), \qquad x(0) = x_0$$

where the control u is given by the elevator command and linearly modifies the moment, lift, and drag coefficients C_M, C_L, C_D. The actual values of parameters A and B are obtained from the identification block. The open-loop model is unstable. The dimension of the model has been chosen $n = 3$. The state of the system is then $x = (e = \theta - \theta_p, \dot{e}, \ddot{e})$. In the criterion (2) we have taken $Q = I_3$, $R = 2$ and the weighting parameter $\lambda = 0.02$.

In Fig. 2 are drawn both a reference trajectory θ_p, with a dotted line, and the attitude angle obtained, with a continuous line. In Fig. 3 is represented the control variable that is the elevator deflection. In the first second the identification has caused a lot of oscillations for the computed control. Moreover, it has been simulated with gust as in Ref 6.

One can see that the system tracks well the reference trajectory (this achieves the performance requirement). The presence of gust does not generate instability (this suggests the achievement of the robustness properties). The value of λ has been chosen as above in order to obtain the same order for the two terms C_P and C_R in the criterion (9).

As a conclusion, the controller tracks the desired trajectory and achieves a trade-off between the performance requirements and the robustness of the stability.

V. Conclusions

In this work an on-line solution for a robust stabilization problem is presented. As an application, a SAS for an aircraft is realized and the results are discussed. The idea behind the robustness criterion is to place the poles of the closed-loop system within some disk included in the unit disk such that the criterion is minimized. The freedom degrees are in this case the radius and the position of the center of the disk. Since α must be a real number, one can see that only symmetric disks with respect to the real axis are allowed.

The criterion to be minimized has two parts [see Eq. (9)], one involving some performance requirements, related to the stabilizable solution of a certain modified DARE [Eq. (1)], and another giving the robustness of the stabilized system. We stress that the stability robustness part [C_R from Eq. (8)] is given by the norm of a state vector and not by the norm of a transfer matrix. This part of the criterion has been chosen like this because of the adaptivity of the solution. The algorithm presented here to solve the optimization problem uses the fixed-point method in order to solve the modified DARE. This fact allows an on-line implementation with a faster adaptivity.

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Matched Asymptotic Expansion Solutions for an Ablating Hypervelocity Projectile

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Introduction

PREVIOUS studies have demonstrated the importance of ablative mass losses of direct launch system projectiles during atmospheric ascent.[1,2] In this Note matched asymptotic expansion solutions[3,4] are obtained which represent the trajectories of such projectiles and incorporate a simple ablation model. Ablative mass losses are parametrically tied to changes in projectile cross section and so are strongly coupled to the projectile flight dynamics. The ablation model assumes that a fraction of the kinetic heating is directed to vaporize the projectile surface. More complex models are available which include heat transfer blocking due to the ablating surface material. However, these effects are found to be small. The solutions obtained give uniformly valid representations of the projectile trajectory from launch tube exit to exoatmospheric Keplerian motion. Such solutions are of interest for the rapid evaluation of direct launch system performance and for other ablative problems, such as kinetic energy antiballistic missile projectiles.

Projectile Dynamics

A generic trajectory geometry will be considered as illustrated in Fig. 1. It will be assumed that there are no transverse forces so that the projectile motion is purely planar. In a geocentric inertial frame the dynamical equations may be written as

$$\frac{dv}{dt} = -\frac{1}{2}\rho C_D \frac{S}{m} v^2 - \frac{\mu}{r^2}\sin\gamma \tag{1a}$$

$$v\frac{d\gamma}{dt} = -\left\{\frac{v^2}{r} - \frac{\mu}{r^2}\right\}\cos\gamma \tag{1b}$$

$$\frac{dr}{dt} = v\sin\gamma \tag{1c}$$

$$\frac{dm}{dt} = -\frac{1}{2}\frac{\kappa}{q}\rho v^3 \tag{1d}$$

where the projectile is characterized by mass m, cross section S, and drag coefficient C_D. The ablative mass loss is modeled by Eq. (1d) by assuming a fraction κ of the kinetic heating is directed to vaporize the projectile surface with specific heat of vaporization q (Ref. 5). Additional terms may be included to model the reduction of heat transfer through the boundary layer. However, for turbulent boundary flow these terms are found to be small.

As the projectile ablates mass, its cross section S will also change. The change in cross section will be described through the shaping parameter α by a mass power law, viz.,

$$S = S_0\{m/m_0\}^\alpha \tag{2}$$

where the projectile has initial mass m_0 and initial cross section S_0. Projectile geometries may be modeled by a suitable choice of shaping parameter. If the ablating projectile remains self-similar then $\alpha = 2/3$, whereas for a cylinder ablating from one end $\alpha = 0$. In

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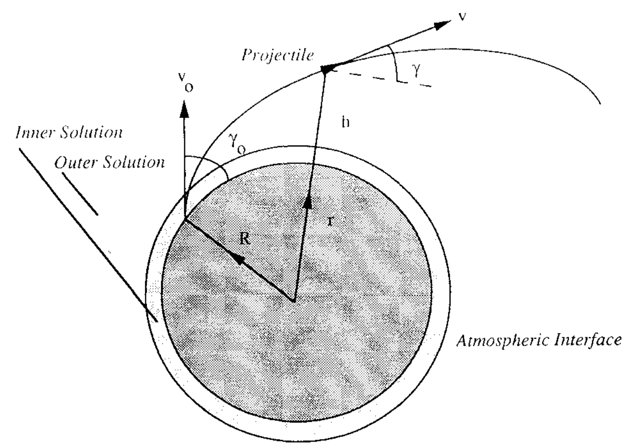

Fig. 1 Schematic trajectory geometry.

principle deformations of the ablating material may also be modeled with $\alpha < 0$ to represent an increased cross section.

The atmosphere is assumed to be exponential with scale height H (6.7 km) and base density ρ_0 (1.752 kgm^{-3}) so that the density has a functional form

$$\rho(r) = \rho_0 \exp\left\{-\frac{1}{H}(r-R)\right\} \tag{3}$$

where R is the mean equatorial radius of the Earth (6371 km).

The projectile dynamical equations may now be transformed to dimensionless form using a new set of state variables, viz.,

$$\bar{v} = v\sqrt{\frac{R}{\mu}}, \qquad h = \frac{r-R}{R}, \qquad \bar{\rho} = \frac{\rho H S_0}{m_0}$$
$$\bar{m} = \frac{m}{m_0}, \qquad \lambda = \frac{\kappa\mu}{2qR}, \qquad \varepsilon = \frac{H}{R} \tag{4}$$

where ε is the small parameter to be used in the asymptotic expansions. Using h as the new independent variable the order of the system is reduced such that

$$\frac{d\bar{v}^2}{dh} = -\frac{\bar{v}^2}{\varepsilon \sin\gamma} C_D \bar{m}^{\alpha-1}\bar{\rho} - \frac{2}{(1+h)^2} \tag{5a}$$

$$\frac{d\gamma}{dh} = \left\{\frac{1}{1+h} - \frac{1}{(1+h)^2\bar{v}^2}\right\}\cot\gamma \tag{5b}$$

$$\frac{d\bar{m}}{dh} = -\frac{\lambda}{\varepsilon \sin\gamma}\bar{m}^{\alpha}\bar{\rho}\bar{v}^2 \tag{5c}$$

Using this reduced set of dynamical equations the method of matched asymptotic expansions may be used to obtain solutions uniformly valid throughout the entire domain of projectile motion.

Matched Asymptotic Expansion Solutions

In general, analytic solutions to Eqs. (5) do not exist, due to their highly nonlinear form. However, during the initial ascent through the atomsphere aerodynamic drag dominates allowing a closed-form inner solution. Similarly, above the sensible atmosphere the motion is Keplerian giving well-known outer solutions. Matched asymptotic expansion methods[6] allow these inner and outer solutions to be matched in such a way that the composite solution is uniformly valid throughout the entire projectile trajectory. This method has been successfully applied to transatmospheric flight dynamics problems in a number of previous studies (see, for example, Ref. 7).

Inner Solution

First, the inner solution of the reduced system of equations will be obtained by repeated application of the inner limit. To facilitate the solution in this region a new, stretched inner variable $\tilde{h} = h/\varepsilon$ will be used along with auxiliary variables $u = v^2$ and $w = \cos\gamma$. The transformed dynamical equations become

$$\frac{du}{d\tilde{h}} = -\frac{u}{\sqrt{1-w^2}} C_D \bar{m}^{\alpha-1}\bar{\rho}_0 \exp\{-\tilde{h}\} - \frac{2}{\{1+\varepsilon\tilde{h}\}^2} \tag{6a}$$

$$\frac{dw}{d\tilde{h}} = -\varepsilon\left\{\frac{1}{(1+\varepsilon\tilde{h})} - \frac{1}{(1+\varepsilon\tilde{h})^2 u}\right\} w \tag{6b}$$

$$\frac{d\bar{m}}{d\tilde{h}} = -\frac{\lambda}{\sqrt{1-w^2}}\bar{m}^{\alpha}\bar{\rho}_0 \exp\{-\tilde{h}\}u \tag{6c}$$

Series solutions to Eqs. (6) are now sought in ascending powers of the small parameter ε, viz.,

$$u(\tilde{h},\varepsilon) = \tilde{u}_0(\tilde{h}) + \varepsilon\tilde{u}_1(\tilde{h}) + \varepsilon^2\tilde{u}_2(\tilde{h}) + \cdots \tag{7a}$$

$$w(\tilde{h},\varepsilon) = \tilde{w}_0(\tilde{h}) + \varepsilon\tilde{w}_1(\tilde{h}) + \varepsilon^2\tilde{w}_2(\tilde{h}) + \cdots \tag{7b}$$

$$\bar{m}(\tilde{h},\varepsilon) = \tilde{m}_0(\tilde{h}) + \varepsilon\tilde{m}_1(\tilde{h}) + \varepsilon^2\tilde{m}_2(\tilde{h}) + \cdots \tag{7c}$$

Substituting these solutions in Eqs. (6) the system of dynamical equations are separated in ascending order of the small parameter ε. To lowest order it is found that

$$\frac{d\tilde{u}_0}{d\tilde{h}} = -\frac{\tilde{u}_0}{\sqrt{1-\tilde{w}_0^2}} C_D \tilde{m}_0^{\alpha-1}\bar{\rho}_0 \exp\{-\tilde{h}\} \tag{8a}$$

$$\frac{d\tilde{w}_0}{d\tilde{h}} = 0 \tag{8b}$$

$$\frac{d\tilde{m}_0}{d\tilde{h}} = -\frac{\lambda}{\sqrt{1-\tilde{w}_0^2}}\tilde{m}_0^{\alpha}\bar{\rho}_0 \exp\{-\tilde{h}\}\tilde{u}_0 \tag{8c}$$

This system of zero-order equations may be integrated to give the inner solution to lowest order as

$$Ei\{\eta\tilde{u}_0\} = Ei\{\eta\tilde{u}_{00}\} + \frac{\tilde{m}_{00}^{\alpha-1}C_D}{\sqrt{1-\tilde{w}_{00}^2}}\bar{\rho}_0\{\exp\{-\tilde{h}\} - \exp\{-\tilde{h}_0\}\} \tag{9a}$$

$$\tilde{w}_0 = \tilde{w}_{00} \tag{9b}$$

$$\tilde{m}_0 = \tilde{m}_{00}\exp\left\{\frac{\eta}{1-\alpha}\tilde{u}_0\right\} \tag{9c}$$

where $\eta = (1-\alpha)\lambda/C_D$ and Ei is the exponential integral function. The inner velocity solution $\tilde{u}_0$ cannot be obtained explicitly as Ei^{-1} does not exist as a combination of elementary or special functions. However, an inverse series for Ei may be easily generated using symbolic mathematics packages.

Outer Solution

The outer solution may now be obtained by applying the outer limit, with h and all other nondimensional variables held constant. Again, a series solution in ascending powers of the small parameter ε is sought, viz.,

$$u(h,\varepsilon) = u_0(h) + \varepsilon u_1(h) + \varepsilon^2 u_2(h) + \cdots \tag{10a}$$

$$w(h,\varepsilon) = w_0(h) + \varepsilon w_1(h) + \varepsilon^2 w_2(h) + \cdots \tag{10b}$$

$$\bar{m}(h,\varepsilon) = m_0(h) + \varepsilon m_1(h) + \varepsilon^2 m_2(h) + \cdots \tag{10c}$$

Substituting these series solutions in Eqs. (5) it is found that to the lowest order in ε

$$\frac{du_0}{dh} = -\frac{2}{(1+h)^2} \tag{11a}$$

$$\frac{dw_0}{dh} = -\left\{\frac{1}{1+h} - \frac{1}{(1+h)^2 u_0}\right\} w_0 \tag{11b}$$

$$\frac{dm_0}{dh} = 0 \tag{11c}$$

where the higher order terms in the asymptotic expansion are identically zero as the zero-order terms give an exact representation of the exoatmospheric Keplerian motion. Equations (11) may now be integrated to give the exact outer solution as

$$u_0 = u_{00} + \frac{2}{1+h} \tag{12a}$$

$$w_0 = \frac{w_{00}}{\sqrt{u_{00}(1+h)^2 + 2(1+h)}} \tag{12b}$$

$$m_0 = m_{00} \tag{12c}$$

Clearly the projectile mass is constant during exoatmospheric flight as expected.

To now determine the integration constants of the outer solution, u_{00}, w_{00}, and m_{00}, an asymptotic matching procedure must now be used. Matching the inner and outer solutions, the unknown outer-solution integration constants are obtained in terms of the inner-solution constants. The inner constants may then be related to the projectile initial conditions.

Composite Solution

Matching the inner and outer solution constants,[3,4] it is found that the uniformly valid composite solution representing ablative projectile motion may now be written to lowest order as

$$u(h) = \eta^{-1} Ei^{-1}\left\{ Ei\{\eta\tilde{u}_{00}\} + \frac{\tilde{m}_{00}^{\alpha-1} C_D}{\sqrt{1-\tilde{w}_{00}^2}} \times \bar{\rho}_0\{e^{-h/\varepsilon} - e^{-h_0/\varepsilon}\}\right\} - \frac{2h}{1+h} \tag{13a}$$

$$w(h) = \tilde{w}_{00}\frac{\sqrt{u_{00}+2}}{\sqrt{u_{00}(1+h)^2 + 2(1+h)}} \tag{13b}$$

$$\bar{m}(h) = \exp\left\{ -\frac{\eta}{1-\alpha}[\tilde{u}_{00} - \tilde{u}_0(h)]\right\} \tag{13c}$$

where the inner velocity solution $\tilde{u}_0$ is obtained from Eq. (9a). These solutions are uniformly valid in the entire domain of projectile motion and represent an ablating ballistic projectile trajectory to lowest order in ε.

Implementation

The composite solutions defined by Eqs. (13) will now be used to compute the trajectory of a hypervelocity ablating projectile. As a numerical example, a projectile fired from sea level with a launch tube elevation of 45 deg will be considered. For ease of illustration the projectile parameters have been chosen ($\eta = 2.56$ and $C_D = 0.1$) to induce rapid ablation of the projectile during its atomspheric pass. The velocity profile shown in Fig. 2 shows rapid drag loss during ascent through the lower atmosphere. On exit from the atmosphere the projectile follows a ballistic Keplerian arc, as expected. Similarly, the projectile mass falls rapidly as material is thermally ablated in the lower atmosphere, Fig. 2. On exit from the atmosphere the projectile has a constant residual mass.

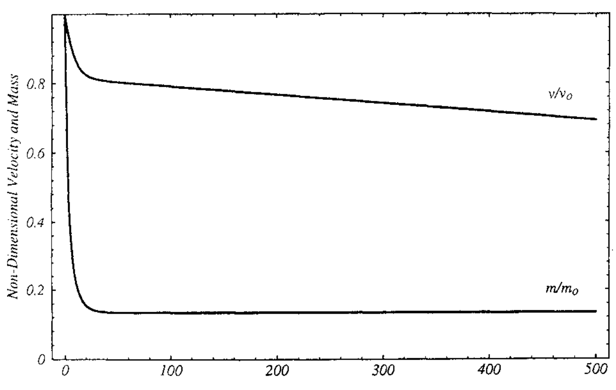

Fig. 2 Projectile velocity and mass profile.

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Approximate Solution to Lawden's Problem

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Introduction

LAWDEN[1] first solved the problem of optimally transferring between two coplanar ellipses whose size and shape are identical but whose lines of apsides are not aligned. Marchal[2] showed that a two-impulse transfer is optimum when the eccentricity is less than 0.535. Lawden's solution provided the true anomaly of the impulse point and the impulse direction of the two symmetric impulses. Unfortunately, his solution required the iterative satisfaction of six simultaneous equations. These equations do not readily provide insight into the effect of eccentricity, say, on the impulse magnitude. Other studies have attempted to overcome this drawback by finding approximate solutions using various simplifying assumptions. For example, Bender[3] and Kuzmak[4] both obtained simpler solutions by assuming that the two impulses are separated by 180 deg. Karrenberg[5] solved for the impulse location and magnitude by assuming that the impulse direction is tangential. In addition, the solutions of both Kuzmak and Karrenberg are restricted to low-eccentricity orbits.

Many three-axis stabilized spacecraft employ a vehicle-fixed coordinate frame where one axis points toward Earth's center. A second axis, orthogonal to the first and lying in the orbit plane, usually defines the thruster pointing direction. From a practical viewpoint, then, it would be useful to have a solution to Lawden's problem where the impulse direction is circumferential and where the results are obtained explicitly rather than iteratively.

Analysis

Given an ellipse of semimajor axis a and eccentricity e, the problem is to rotate the line of apsides through an angle $\Delta\omega$. Following Lawden, assume that the transfer is accomplished by two symmetrical, equal-magnitude impulses, each of which rotates the line of

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